AN EXPLORATION OF GAMES

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ABSTRACT. We examine two-player, sequential, multi-battle, fixed-budget contests with majoritarian objective. Players have fixed, integer-valued budgets and must allocate these budgets to a sequence of castles, where each castle has an associated value. The outcome of each battle is a function of the budget allocated to that castle. The winner of the contest is the player who wins the castles with highest total value.

1. Introduction

The Colonel Blotto game was proposed by Borel in 1921 [Bor53]. In this game, two players fight on n simultaneous and independent castles of different values. Each player has a finite budget of soldiers, B, to split over the castles, and for each castles k, the value w_k, will be awarded to the player that allocates more power there. Each player aims to win the game. The game is called symmetric if all players have the same budget, homogenous if all castles have the same value, continuous if allocations can be arbitrary positive real numbers, and discrete if allocations are restricted to be whole numbers [Jay21]. In this project, we consider the 2-person, symmetric, discrete Colonel Blotto game. Moreover, for each $k = 1, 2, \ldots, n$, we let $w_k = k$.

We are motivated by two primary research questions. Given a fraction of allocations $t = \frac{p}{B}$, what is the probability $P(t)$ that Player 1 wins? How might we construct a player that will learn to maximize its chances of winning? Our main results are as follows.

Theorem. *Assume that we have a total of n castles and b soldiers. For each $k = 1, 2, \ldots, n$, we denote*

- *p_k as the number of soldiers placed in castle k by Player 1,*
- *q_k as the number of soldiers placed in castle k by Player 2,*
- *w_k as the number of points won for having more soldiers in castle k, and*
- *$N(p)$ as the number of ways for Player 1 to win the game with strategy p.*

Then, we have the following:

(1) Player 1 has $N = \binom{B+n-1}{n-1} = \frac{(B+n-1)!}{B!(n-1)!}$ possible strategies

(2) Player 1 wins the game if and only if the expression

$$Z(p, q) = \sum_{k=1}^{n} w_k h(p_k - q_k) = \sum_{k=1}^{n-1} w_k h(p_k - q_k) - w_n h((p_1 - q_1) + \cdots + (p_{n-1} - q_{n-1}))$$

is positive, expressed in terms of the function

$$h(x, y) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0, \text{ and} \\ -1, & \text{if } x < 0. \end{cases}$$

(3) $P(t) = \frac{N_p}{N}$ is the probability that Player 1 wins the game with allocation p.

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2. Background and Notation

We begin by introducing definitions and known results relevant for our main research questions.

2.1. Colonel Blotto. The Colonel Blotto game is specified by a tuple

$$\left(k \in \mathbb{N}, n \in \mathbb{N}, \vec{\mathcal{B}} \in \mathbb{Z}^n_{\geq 0}, \vec{v} \in \mathbb{Z}^n_{\geq 0}\right),$$

where k is the number of players, n is the number of castles, $\mathcal{B}_i$ is the budget of player $i \in [k]$, and v_j is the value of castle $j \in [n]$. We denote the sum total of the castle values by $V = \|\vec{v}\|_1 = \sum_{j=1}^{n} v_j$.
Each player $i \in [k]$ plays a bid vector $A_{i,*} = (A_{i,1}, \ldots, A_{i,n}) \in \mathbb{Z}^n_{\geq 0}$ satisfying the budget constraint

$$\|A_{i,*}\|_1 = \sum_{j \in [n]} A_{i,j} = \mathcal{B}_i$$

.
The game is called symmetric if all the player budgets are equal, and homogeneous if all castle values are equal. [Jay21]

3. Main Results

In this section, we list the main results from our research.

Consider the Colonel Blotto game with equal budgets and two castles.

	C1	C2
Player 1	p_1	p_2
Player 2	q_1	q_2

Motivating Question. *What has to happen for Player 1 to win?*

Proposition 1. *Say that $p = (p_1, p_2)$ is the placement of the soldiers for Player 1, and $q = (q_1, q_2)$ is the placement of the soldiers by Player 2. We assume that $w_1 < w_2$, such as with $(w_1, w_2) = (1, 2)$. Then, Player 1 wins if and only if $p_1 < q_1$.*

Proof. If $p_1 - q_1 = 0$, then this means that $p_1 = q_1$ and $p_2 = q_2$. Neither player will receive any points. Hence, the game will result in a tie. If $p_1 - q_1 < 0$, then this means that $p_1 < q_1$ and $p_2 > q_2$. Player 1 will receive w_2 points and Player 2 will receive w_1 points. Hence, Player 1 will win the game. If $p_1 - q_1 > 0$, then this means that $p_1 > q_1$ and $p_2 < q_2$. Player 1 will receive w_1 points and Player 2 will receive w_2 points. Hence, Player 1 will lose the game. $\square$

Motivating Question. *The total number of strategies is $N = B + 1$. Given a fixed $t = (t_1, t_2)$ with $0 \leq t_k \leq 1$ and $t_1 + t_2 = 1$, let $p_1 = B \cdot t_1$ and $p_2 = B \cdot t_2$. How many strategies $q = (q_1, q_2)$ are there with $p_1 < q_1$?*

The number of strategies $q = (q_1, q_2)$ with $p_1 < q_1$ is $|\{q_1 \in \mathbb{Z} \mid p_1 < q_1 \leq B\}| = B - p_1 = B(1 - t_1) = B\dot{t}_2$. This implies that $P(t) = \frac{B}{B+1}(1 - t_1) = \frac{B}{B+1}t_2$.

Now, consider the Colonel Blotto game with equal budgets and three castles. Say that $p = (p_1, p_2, p_3)$ is the placement of the soldiers for Player 1, and $q = (q_1, q_2, q_3)$ is the placement of the soldiers by Player 2.

	C1	C2	C3
Player 1	p_1	p_2	p_3
Player 2	q_1	q_2	q_3

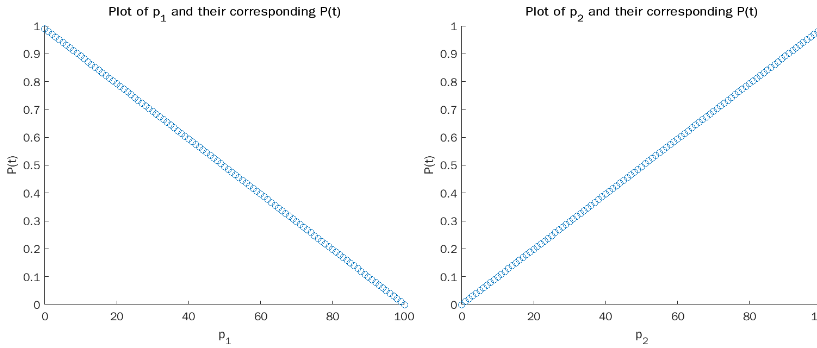

FIGURE 1. In both plots, we consider the Colonel Blotto game with 100 troops and 2 castles. On the left is a plot of Player 1's probability of winning the game based on their allocation to castle 1. On the right, a plot of Player 1's probability of winning the game based on their allocation to castle 2.

Motivating Question. *What has to happen for Player 1 to win?*

Proposition 2. *We assume that $w_1 < w_2$ and $w_3 = w_1 + w_2$, such as with $(w_1, w_2, w_3) = (1, 2, 3)$. Player 1 wins if and only if*

- $p_1 < q_1$ *and* $p_2 = q_2$
- $p_1 < q_1$, $p_2 > q_2$, *and* $p_3 \geq q_3$
- $p_1 = q_1$ *and* $p_2 < q_2$
- $p_1 > q_1$, $p_2 < q_2$, *and* $p_3 > q_3$

Proof. We define a helper function $f(x, y)$ where

$$f(x,y) = \begin{cases} 1, & \text{if } x > y \\ 0, & \text{otherwise.} \end{cases}$$

Let $F(p_1, p_2, q_1, q_2) = w_1 \cdot f(p_1, q_1) + w_2 \cdot f(p_2, q_2) + w_3 \cdot f(B - p_1 - p_2, B - p_1 - p_2)$ and $G(p_1, p_2, q_1, q_2) = w_1 \cdot f(q_1, p_1) + w_2 \cdot f(q_2, p_2) + w_3 \cdot f(B - q_1 - q_2, B - p_1 - p_2)$ denote the total number of points that Player 1 and Player 2 wins, respectively. We want to know when $F > G$. We must first ask: how does $f(x, y)$ compare to $f(y, x)$ for some values $x, y \in \mathbb{Z}_{\geq 0}$? In other words, what is $f(x, y) - f(y, x)$? We consider three cases: $x = y$, $x < y$, and $x > y$. If $x = y$, then $f(x, y) = 0$ and $f(y, x) = 0$. Hence, $f(x, y) - f(y, x) = 0 - 0 = 0$. If $x < y$, then $f(x, y) = 0$ and $f(y, x) = 1$. Hence, $f(x, y) - f(y, x) = 0 - 1 = -1$. If $x > y$, then $f(x, y) = 1$ and $f(y, x) = 0$. Hence, $f(x, y) - f(y, x) = 1 - 0 = 1$. So, the function

$$F - G = w_1 \cdot (f(p_1, q_1) - f(q_1, p_1)) + w_2 \cdot (f(p_2, q_2) - f(q_2, p_2)) + w_3 \cdot (f(B - p_1 - p_2, B - q_1 - q_2) - f(B - q_1 - q_2, B - p_1 - p_2))$$

is equivalent to the function $h(x, y)$, where

$$h(x,y) = \begin{cases} 1, & \text{if } x > y \\ 0, & \text{if } x = y \\ -1, & \text{if } x < y. \end{cases}$$

Observe that $h(B - x, B - y) = -f(x, y) = +f(y, x)$.

We define a new function Z, where

$$\begin{aligned}
Z(p_1,p_2,q_1,q_2) &= w_1 \cdot h(p_1,q_1) + w_2 \cdot h(p_2,q_2) + w_3 \cdot h(B-p_1-p_2, B-q_1-q_2) \\
&= \sum_{k=1}^{n-1} w_k \cdot h(p_k,q_k) + w_n h(B-p_1-\cdots-p_{n-1}, B-q_1-\cdots-q_{n-1}) \\
&= \sum_{k=1}^{n-1} w_k \cdot h(p_k,q_k) - w_n h(p_1+\cdots+p_{n-1}, q_1+\cdots+q_{n-1})
\end{aligned}$$

Player 1 wins if and only if $Z > 0$. We want to find when $Z > 0$. We have multiple cases:

	$p_1 > q_1$	$p_1 = q_1$	$p_1 < q_1$
$p_2 > q_2$	$\vec{Z} = (1,1,-1)$ $Z = w_1 + w_2 - w_3$ Tie	$\vec{Z} = (0,1,-1)$ $Z = w_2 - w_3$ Player 2 wins	$\vec{Z} = (-1,1,\text{Unk})$ Z = Unk
$p_2 = q_2$	$\vec{Z} = (1,0,-1)$ $Z = w_1 - w_3$ Player 2 wins	$\vec{Z} = (0,0,0)$ $Z = 0$ Tie	$\vec{Z} = (-1,0,1)$ $Z = -w_1 + w_3$ Player 1 wins
$p_2 < q_2$	$\vec{Z} = (1,-1,\text{Unk})$ Z = Unk	$\vec{Z} = (0,-1,1)$ $Z = -w_2 + w_3$ Player 1 wins	$\vec{Z} = (-1,-1,1)$ $Z = -w_1 - w_2 + w_3$ Tie

	$p_1+p_2 > q_1+q_2$	$p_1+p_2 = q_1+q_2$	$p_1+p_2 < q_1+q_2$
$p_1 < q_1 \wedge p_2 > q_2$	$\vec{Z} = (-1,1,-1)$ Z $= -2w_1$ Player 2 wins	$\vec{Z} = (-1,1,0)$ Z $= w_2 - w_1$ Player 1 wins	$\vec{Z} = (-1,1,1)$ Z $= 2w_2$ Player 1 wins
$p_1 > q_1 \wedge p_2 < q_2$	$\vec{Z} = (1,-1,-1)$ Z $= -2w_2$ Player 2 wins	$\vec{Z} = (1,-1,0)$ Z $= w_1 - w_2$ Player 2 wins	$\vec{Z} = (1,-1,1)$ Z $= 2w_1$ Player 1 wins

□

Motivating Question. *The total number of strategies is* $N = \frac{(B+1)(B+2)}{2}$. *Given a fixed* $t = (t_1,t_2,t_3)$ *with* $0 \le t_k \le 1$ *and* $t_1+t_2+t_3 = 1$, *let* $p_1 = B \cdot t_1$, $p_2 = B \cdot t_2$, *and* $p_3 = B \cdot t_3$. *How many strategies* $q = (q_1,q_2,q_3)$ *are there that satisfy Proposition 2?*

The number of strategies $q = (q_1,q_2,q_3)$ that satisfy Proposition 2 is

$$\begin{aligned}
&= \{q = (q_1,q_2,q_3) \mid p_1 < q_1, p_2 = q_2 \vee p_1 < q_1, p_2 > q_2, p_3 \ge q_3 \vee p_1 = q_1, p_2 < q_2 \vee p_1 > q_1, p_2 < q_2, p_3 > q_3\} \\
&= \{q = (q_1,q_2,q_3) \mid 0 < q_1 \le B - p_2, q_2 = p_2, q_3 = B - q_1 - q_2\} \\
&+ \{q = (q_1,q_2,q_3) \mid B - q_2 - p_3 \le q_1 \le B - q_2, 0 \le q_2 < p_2, q_3 = B - q_1 - q_2\} \\
&+ \{q = (q_1,q_2,q_3) \mid q_1 = p_1, p_2 < q_2 \le B - p_1, q_3 = B - q_1 - q_2\} \\
&+ \{q = (q_1,q_2,q_3) \mid 0 \le q_1 < p_1, B - q_1 - p_3 < q_2 \le B - q_1, q_3 = B - q_1 - q_2\} \\
&= p_3 + p_2(p_3+1) + p_3 + p_1 p_3 \\
&= Bt_3 + Bt_2(Bt_3+1) + Bt_3 + B^2 t_1 t_3
\end{aligned}$$

This implies that $P(t) = \frac{Bt_3 + Bt_2(Bt_3+1) + Bt_3 + B^2 t_1 t_3}{\frac{(B+1)(B+2)}{2}}$. Moreover, $\lim_{B\to\infty} P(t) = \frac{t_2 t_3 + t_1 t_3}{\frac{1}{2}} = 2t_3(1-t_3)$.

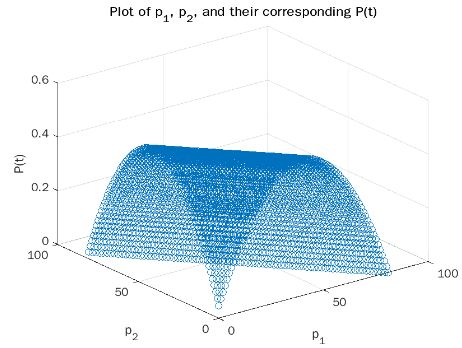

FIGURE 2. A plot of Player 1's probability of winning a Colonel Blotto game with 100 soldiers and 3 castles based on their allocations to castle 1 and 2.

Now, we consider how a Q-Learning agent performs against a Random agent in Colonel Blotto by reproducing results by Noel [Noe22] and, at the same time, we should realize the work above in our results. We set up a Colonel Blotto game with 3 fields and 100 soldiers per player. This provides a total of 5,151 possible actions, or strategies. The winner of the game will receive a reward of $r = 1$, the loser will receive a reward of $r = -1$, and no reward is received in the event of a tie. A single game constitutes a single episode in reinforcement learning. We count how many games each player has won over time and show the results. Our reinforcement learning agent uses the Q-Learning algorithm for approximating the optimal policy. We use $\alpha = 0.1$ as the learning rate and $\gamma = 1$ as the discount factor. We also set $\epsilon = 0.2$ as the exploration rate for the agent. For the opponent, we use a Random agent that will select one of the 5151 possible actions at random with equal probability for all of them. The experiments were run using the OpenSpiel [Lan+19] environment simulator. We run 100,000 games of Colonel Blotto to see the performance of the reinforcement learning agent and show the results in Figure 1.

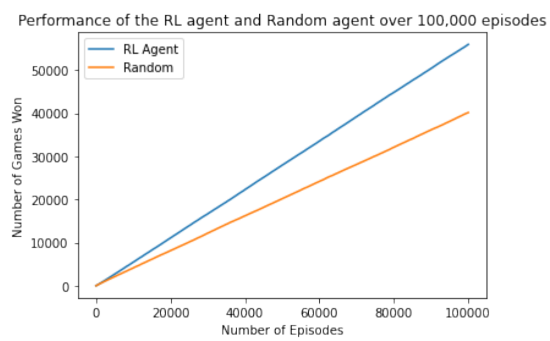

FIGURE 3. A plot of the number of games won over 100,000 episodes for both the RL and Random agent.

We also dive deeper and look at the Q-values that the agent has assigned to each of the 5,151 possible actions, to better understand what strategy it has arrived at. Below we show the top 10 and the bottom 10 actions based on their Q-value scores.

As can be seen, the actions with the highest and lowest Q-values follow from our main results.

4. Ethics

There are no ethics to discuss here. As it stands, we are merely playing games. The ethics to be discussed may lie in the applications of this work. For example, consider an election. Each candidate must decide how to allocate their resources across districts [Jay21]. Each candidate would want to know how to allocate their resources optimally so as to win the election. This becomes more troublesome when the application is war.

Top 10 Actions	Bottom 10 Actions
(44,26,30)	(0,1,99)
(34,31,35)	(4,94,2)
(68,20,12)	(6,88,6)
(3,48,49)	(8,1,91)
(37,57,6)	(77,15,8)
(25,35,40)	(84,13,3)
(47,41,12)	(96,2,2)
(64,31,5)	(81,6,13)
(31,0,69)	(5,73,22)
(33,21,46)	(19,1,80)

5. Reflections

For next time, we will provide a more elegant project by engaging tightly with the literature.

6. Future Work

We would like to characterize the Colonel Blotto game with four castles and ultimately generalize to any number of castles. We would also like to engage with more literature. This would be for the sake of familiarizing ourselves with terminology, formalizing our own work, and furthering our understanding of the ideas of others already in this space. Moreover, this would allow us to explore connections from game theory to information theory and psychology.

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