Phenomenon of Collinearity

Tesfa Asmara and Edray Herber Goins

Say (E, β) is a Toroidal Belyĭ pair, and denote $\Gamma = \beta^{-1}\big(\{0,\, 1,\, \infty\}\big)$ as the collection of quasi-critical points. If Γ is a group, then we have a collinearity condition: given two points $P,\, Q \in \Gamma$, we have $P \oplus Q \in \Gamma$ as well, so the point $R = [-1]\,(P \oplus Q)$ which lies on a line through P and Q must also be in Γ.

We can ask if there is a collinearity condition that will still hold when Γ is not a group. Consider the Toroidal Belyĭ pair (E, β) with E defined by $f(x, y) = y^2 - (x^3 + x^2 + 16)$ and $\beta(x, y) = (4y + x^2 + 56)/108$. We can compute the critical points $P = (x, y)$ of β by finding when the following function vanishes: $(\partial f/\partial x)\,(\partial \beta/\partial y) - (\partial f/\partial y)\,(\partial \beta/\partial x) = (-3x^2 - 2x - yx)/27$. We find that the critical points are $\{(4, -18), (-2, 12), (22, -108), O_E\}$. Observe that $P = (4, -18) \in \Gamma$. However, $-P = (4, 18) \notin \Gamma$. Thus, Γ is not a group. Yet, we see that $P = (4, -18), Q = (22, -108), R = (22, -108) \in \Gamma$ and that these three quasi-critical points lie on a lie.

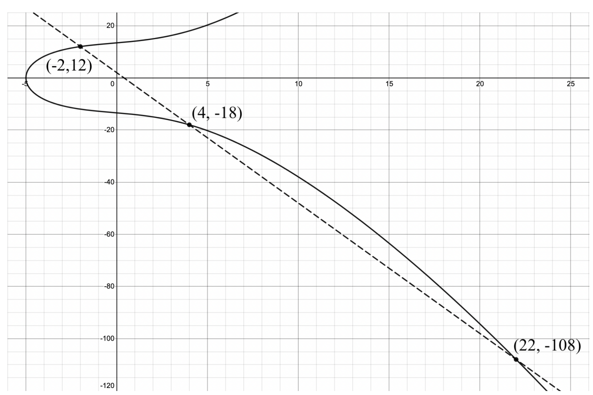

Figure 1. $(4, -18)$, $(-2, 12)$, $(22, -108)$ on $E : y^2 = x^3 + x^2 + 16\,x + 180$

Research Question. *Say (E, β) is a Toroidal Belyĭ pair, and denote $\Gamma = \beta^{-1}\big(\{0,\, 1,\, \infty\}\big)$ as the collection of quasi-critical points. When does Γ have three points P, Q, R that lie on a line?*

The outline of our methodology is as follows: Given $\Gamma = \beta^{-1}\big(\{0,\, 1,\, \infty\}\big)$ as the collection of quasi-critical points associated to a Toroidal Belyĭ pair (E, β), find all subsets of three distinct points in Γ and record those Belyĭ pairs that have subsets of points that lie on a line. Computational power limited our ability to calculate the number field L in many cases. As a result, we implemented a "time-out," a specified time interval for computing the number field after which the example would be skipped. Still we were able to find examples of $\Gamma = \beta^{-1}\big(\{0,\, 1,\, \infty\}\big)$ associated to Toroidal Belyĭ pairs (E, β) from LMFDB, where one or more subsets of points $P, Q, R \in \Gamma$ lie on a line. There are three cases for a subset of points $P, Q, R \in \Gamma$.

- Say that $P = Q = R$. Then this is only true if and only if $[3]P = O_E$. There are a maximum of 9 such points.
- Say that $P \neq Q$ and $P = R$. Then $[2]P \oplus Q = O_E \iff Q = [-2]P$. There are a maximum of N such points for a elliptic curve of order N.
- Say that P, Q, R are distinct. Then there are $\frac{n(n-1)(n-2)}{6} = \binom{n}{3}$ such points.

Pomona College, 610 North College Avenue, Claremont CA 91711
Email address: tgac2020@mymail.pomona.edu

Pomona College, 610 North College Avenue, Claremont CA 91711
Email address: edray.goins@pomona.edu