Abelian Divison Fields Over Real Quadratic Fields

Alex Abrams Tesfa Asmara David W. Bonds, Jr. Aniyah Stephen

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Our Project

- We want to look at extension fields that arise from elliptic curves and points on those curves of specified order.
- When are the Galois groups corresponding to these fields abelian?

Outline

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1. Definitions
2. Goal
3. Narrowing Our Search
4. Results

Extension Field

Definition

Given fields K and F, we say K is an **extension field** of F if $F \subseteq K$. We denote this as K/F.

- When you add something to a field, you have to add additional elements to make sure you still have a field.

Examples

$$\mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$$
$$\mathbb{Q}(\sqrt[3]{19}) = \{a + b\sqrt[3]{19} + c(\sqrt[3]{19})^2 \mid a, b, c \in \mathbb{Q}\}$$
$$\mathbb{Q}(i, \sqrt{2}) = \{a + b\sqrt{2} + ci + di\sqrt{2} \mid a, b, c, d \in \mathbb{Q}\}$$

For an extension K/F, we call the field F the **base field**.

Why Extension Fields

Question

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If K/F is a field extension, then K is a vector space over F and the **degree of K over F** is the dimension.

Elliptic Curves

Introduction

An elliptic curve, E, over $\mathbb{Q}$ can be defined by an equation of the form $y^2 = x^3 + Ax + B$, where A,B $\in \mathbb{Q}$ and $\Delta_E = -16(4A^3 + 27B^2) \neq 0$.

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The discriminant, Δ_E, of an elliptic curve E defined over $\mathbb{Q}$, is a nonzero integer (when $\Delta_E \neq 0$, the equation $x^3 + Ax + B$ has distinct roots over the complex numbers). We require $\Delta_E \neq 0$ because it is necessary for the elliptic curve to have a group structure.

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Elliptic Curves have been used in Cryptography and to prove

- Fermat's Last Theorem
- Gauss's Class#1 Conjecture

Group Law on Elliptic Curves

Definition

We denote $E(\mathbb{Q})$ as the set of points (x, y) such that x and y are rational numbers that satisfy $y^2 = x^3 + Ax + B$ along with a point at infinity, denoted O_E.

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Proposition

1. There exists a binary operation $\oplus$ such that $(E(\mathbb{Q}), \oplus)$ is an abelian group with identity O_E.
2. Points P, Q, R on $E(\mathbb{Q})$ lie on a line if and only if $P \oplus Q \oplus R = O_E$.

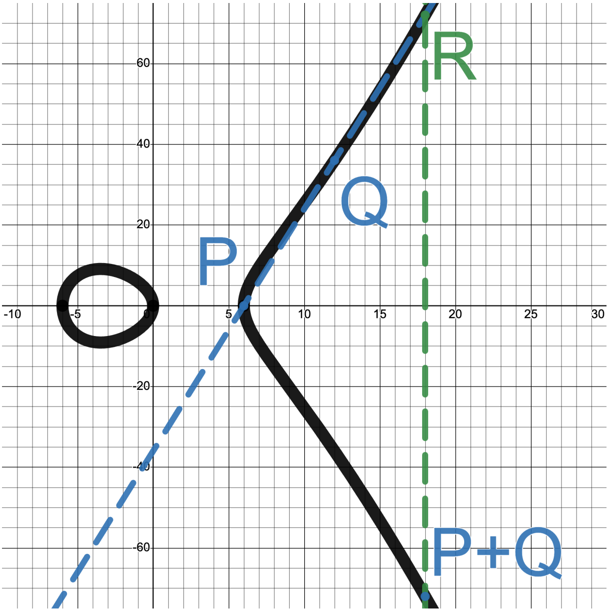

Figure: The Chord-and-Tangent Construction on $E : y^2 = x^3 - 36x$

Torsion Points

Definition

A point $P \in E(\mathbb{Q})$ has **order** n if n is the smallest positive integer such that

$$[n]P = \underbrace{P \oplus P \oplus \cdots \oplus P}_{n \text{ times}} = O_E$$

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If such n exists, P is said to have finite order, otherwise it has infinite order.

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Definition

A point $P \in E(\mathbb{Q})$ is called a **torsion point** if it has finite order.

Demonstration of the Group Law on Torsion Points

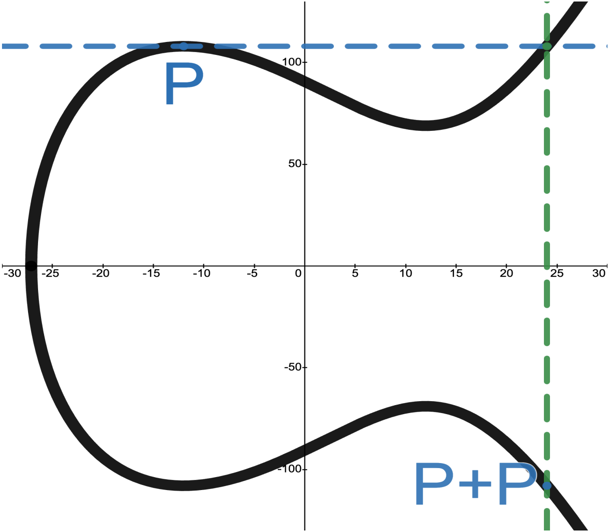

Figure: Computation of P+P on $E : y^2 = x^3 - 432 * x + 8208$

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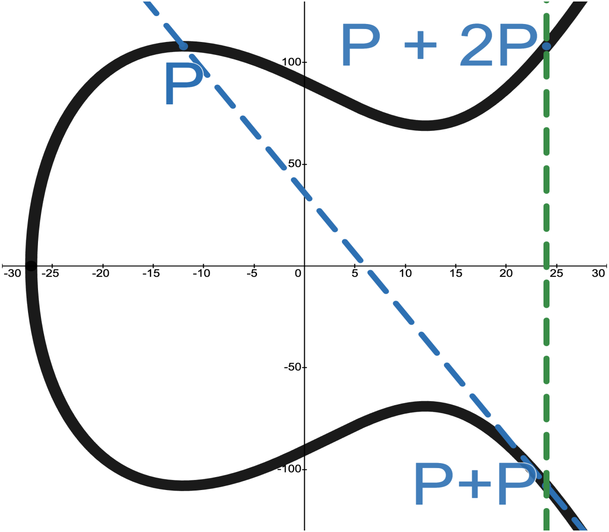

Figure: Computation of P+2P on $E : y^2 = x^3 - 432 * x + 8208$

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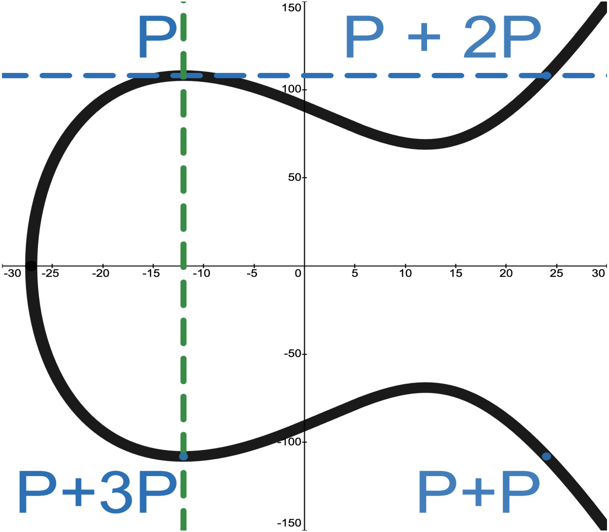

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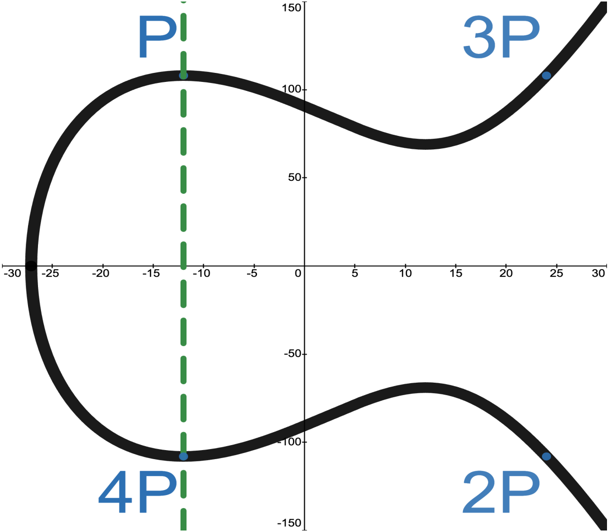


Figure: All of the 5-torsion points and O_E on $E : y^2 = x^3 - 432 * x + 8208$

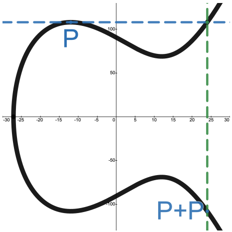
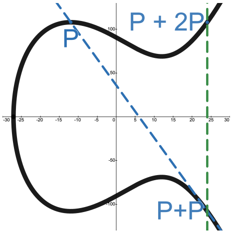


Figure: Computation of P+P



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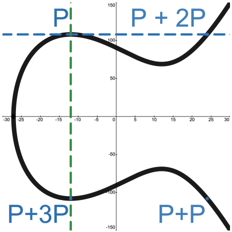

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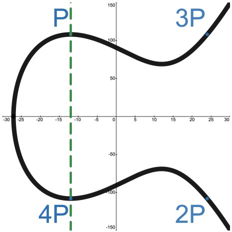

Figure: All 5-torsion points and O_E

Torsion Points

Definition

The set of rational torsion points for an elliptic curve E denoted $E(\mathbb{Q})_{\text{tors}}$.

Theorem (Mordell-Weil, 1928)

The group of rational points on an elliptic curve denoted $E(\mathbb{Q})$ is a finitely generated abelian group.

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \times E(\mathbb{Q})_{\text{tors}}$$

Mazur's Theorem

Theorem (Mazur, 1977-1978)

Let E be an elliptic curve over $\mathbb{Q}$. If $P \in E(\mathbb{Q})_{\text{tors}}$ is a point of prime order p, then $p < 11$.

- This tells us there are no non-trivial rational prime p-torsion points when $p \geq 11$.
- There are also no non-trivial rational n-torsion points when n is a multiple of a prime $p \geq 11$.

Division Fields

Definition

Let E be an elliptic curve. Let K be a field. The **n-th division field of E/K** denoted $K(E[n])/K$ is an extension field of K with all the points of n torsion.

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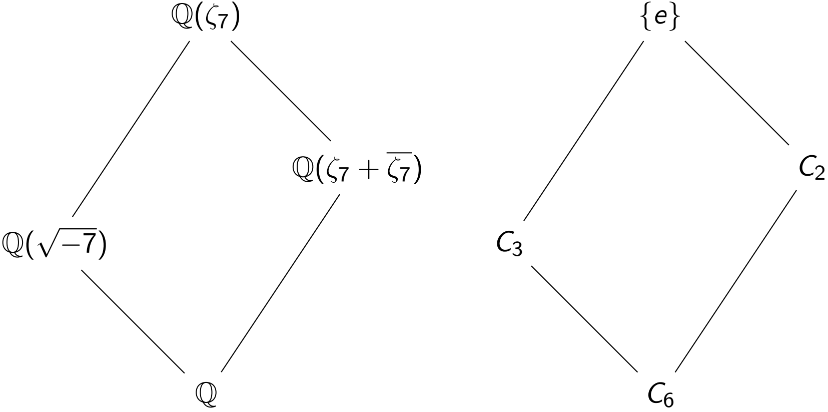
Definition

A division field is **abelian** if its corresponding Galois group is abelian.

By the Fundamental Theorem of Galois Theory, there is a correspondence between the subfield structure of Galois extensions and the structure of the subgroups of the Galois group.



Cyclotomic Fields

Definition

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Definition

The nth cyclotomic field is the extension $\mathbb{Q}(\zeta_n)$.

Weil Pairing

Theorem: Weil-Pairing (Weil, 1940)

Let K be a field, ζ_n be a primitive nth root of unity where $n \in \mathbb{Z}^+$, and E be an elliptic curve over K. Then, $K(\zeta_n) \subseteq K(E[n])$.

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By the Weil-Pairing, for each $n \geq 3$, we know that there is a quadratic extension field F such that $F \subseteq \mathbb{Q}(\zeta_n) \subseteq \mathbb{Q}(\mathrm{E}[n])$.

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We have $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3) \subseteq \mathbb{Q}(\mathrm{E}[3])$

Similarly, we have $\mathbb{Q}(\sqrt{5}) \subseteq \mathbb{Q}(\zeta_5) \subseteq \mathbb{Q}(\mathrm{E}[5])$

as well as, $\mathbb{Q}(\sqrt{-7}) \subseteq \mathbb{Q}(\zeta_7) \subseteq \mathbb{Q}(\mathrm{E}[7])$

Kronecker-Weber

Kronecker-Weber Theorem(Kronecker, 1853; Weber, 1886; Hilbert, 1895)

For a field L, if $L/\mathbb{Q}$ is an abelian extension, then $L \subseteq \mathbb{Q}(\zeta_n)$ where $n \in \mathbb{Z}$.

Previous Work on Division Fields

Theorem (González-Jiménez and Lozano-Robledo, 2017)

In "Elliptic Curves with Abelian Division Fields", Enrique González-Jiménez and Álvaro Lozano-Robledo showed for which curves the n-th division field is as small as possible, meaning that $\mathbb{Q}(\zeta_n) = \mathbb{Q}(E[n])$, which is only possible when $n = 2, 3, 4$, or 5.

Previous Work on Division Fields

Theorem (González-Jiménez and Lozano-Robledo, 2017)

They also determined for which n a field $\mathbb{Q}(E[n])$ is contained in some cyclotomic extension of $\mathbb{Q}$ or, equivalently, when $\mathbb{Q}(E[n])/\mathbb{Q}$ is an abelian extension. This only happens when $n = 2, 3, 4, 5, 6$, or 8.

$\mathrm{GL}_2(\mathbb{F}_p)$

Let $\mathbb{F}_p$ be the field with p elements, where p is prime. The group $\mathrm{GL}_2(\mathbb{F}_p)$ is

$$\mathrm{GL}_2(\mathbb{F}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \not\equiv 0 \pmod{p} \right\}.$$

$\mathrm{GL}_2(\mathbb{F}_p)$

Definition

Let $E[p]$ denote the set of p-torsion points of the elliptic curve E. This means that $E[p] = \{R \in E(\mathbb{Q}) \mid [p]R = O_E\}$.

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The set $E[p]$ of p-torsion points is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. As a result, the set of automorphisms of $E[p]$ is isomorphic to $GL_2(\mathbb{F}_p)$.

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Since each field automorphism of the field $\mathbb{Q}(E[p])$ will also be an automorphism of $E[p]$, $\mathrm{Gal}(\mathbb{Q}(E[p])/\mathbb{Q})$ is isomorphic to a subgroup of $GL_2(\mathbb{F}_p)$.

Serre's Result

Theorem (Serre, 1972)

Let $E/\mathbb{Q}$ be a non-CM elliptic curve. Let $\mathrm{Gal}(\mathbb{Q}(E[p])/\mathbb{Q}) \cong G \subseteq GL_2(\mathbb{F}_p)$, and supppose $G \ncong GL_2(\mathbb{F}_p)$. Then, there is an $\mathbb{F}_p$-basis of $E[p]$ such that one of the following possibilities holds:

(1) G is contained in the normalizer of a split Cartan subgroup of $GL_2(\mathbb{F}_p)$, or

(2) G is contained in the normalizer of a non-split Cartan subgroup of $GL_2(\mathbb{F}_p)$, or

(3) The projective image of G in $PGL_2(\mathbb{F}_p)$ is isomorphic to A_4, S_4, or A_5, where S_n is the symmetric group and A_n is the alternating group, or

(4) G is contained in a Borel subroup of $GL_2(\mathbb{F}_p)$.

Why Serre's Result?

Key Points

- We find that Serre's result is important for our method to find specific Galois groups.

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- We find that Serre's result is important for our method to find specific Galois groups.
- We are only looking at elliptic curves defined over $\mathbb{Q}$ to see when $\mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$ can become abelian over $\mathbb{Q}(\sqrt{5})$.
- Since the Galois groups in question are necessarily contained in certain subgroups of $GL_2(\mathbb{F}_p)$, we can narrow down our search to find what subgroups are even possible to become abelian over $\mathbb{Q}(\sqrt{5})$.

Non-CM Elliptic Curves

- **CM elliptic curves** are a special kind of elliptic curve.
- Serre's result applies to certain curves called **non-CM elliptic curves**.
- Most elliptic curves are non-CM elliptic curves.
- CM elliptic curves are known to have smaller division fields.
- We know what these curves are since there are 13 j-invariants that correspond to CM elliptic curves.

Definition

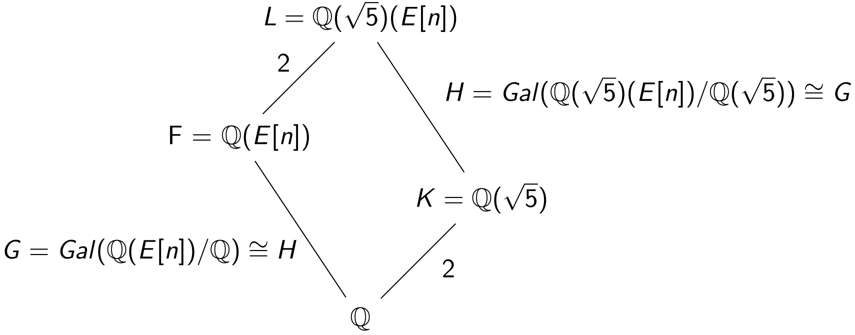
Given an elliptic curve defined by $y^2 = x^3 + Ax + B$, the **j-invariant** can be defined as

$$j(E) = 1728\frac{4A^3}{4A^3 + 27B^2}$$

Outline

Our Question

- When are division fields of non-CM elliptic curves abelian over real quadratic fields?
- Specifically, when is $\mathbb{Q}(\sqrt{5})(E[n])/\mathbb{Q}(\sqrt{5})$ abelian?



General Method

- We want to start with choosing a prime p.
- We consider all the possible Galois groups of p division fields over an arbitrary non-CM elliptic curve.
- We determine whether or not $\mathbb{Q}(\sqrt{5})$ can be contained in these division fields.
- If that division field is not abelian over $\mathbb{Q}$, then we want to see if it is abelian over $\mathbb{Q}(\sqrt{5})$.
- We compute Galois groups and their subgroups to determine more about the fields and their subfields.

Outline

Abelian Division Fields Over $\mathbb{Q}$

Proposition

If $\mathbb{Q}(E[n])/\mathbb{Q}$ is abelian then $\mathbb{Q}(\sqrt{5}, E[n])/\mathbb{Q}(\sqrt{5})$ is abelian.

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- This gives us infinite examples of abelian division fields over $\mathbb{Q}(\sqrt{5})$.
- We are interested when this is not the case.

Good Reduction

Proposition

Let $5 \nmid n$ and $5 \nmid \Delta_E$. If $\mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$ is non-abelian, then $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})(E[n])/\mathbb{Q}(\sqrt{5}))$ is non-abelian as well.

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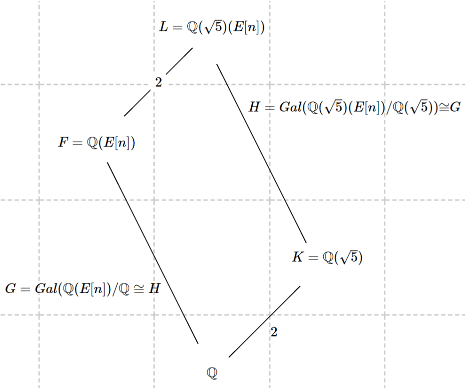
In this case we are assuming that 5 is a prime of good reduction. When 5 is a prime of good reduction and $5 \nmid n$ we know a non-abelian n-division field will remain non-abelian over $\mathbb{Q}(\sqrt{5})$.

Good Reduction



Under these conditions $Gal(L/\mathbb{Q}(\sqrt{5})$ is isomorphic to Gal$(F/\mathbb{Q})$

Bad Reduction

Motivating Question

When can non-abelian over $\mathbb{Q}$ become abelian over $\mathbb{Q}(\sqrt{5})$?

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Motivating Results

- The conditions for having a prime, in our case 5, be of bad reduction is that $5 \mid \Delta_E$.
- In order to have this occur $\sqrt{5}$ must be contained in $\mathbb{Q}(E[n])$ and this can only happen when $5 \mid n$ or $5 \mid \Delta E$. If $5 \mid n$, then by the Weil-Pairing, $\mathbb{Q}(\sqrt{5}) \subset \mathbb{Q}(E[n])$

Prime Division Fields

We have proven this result

Proposition

Let L, C, F be fields such that $F \subseteq C \subseteq L$. Let L/F be Galois, and let C/F be Galois. Then if $\mathrm{Gal}(C/F)$ is not abelian, $\mathrm{Gal}(L/F)$ is not abelian.

Which gives us a useful corollary

Corollary

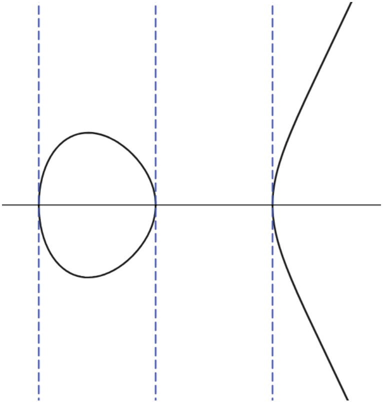
If $K(E[n])/K$ is not abelian, then $K(E[dn])/K$ is not abelian for $d \in \mathbb{Z}^+$. And if $K(E[dn])/K$ is abelian, then $K(E[n])/K$ is abelian.

$$\begin{array}{c} K(E[dn]) \\ | \\ K(E[n]) \\ | \\ K \end{array}$$

Outline

2 Division Fields

- Let E be the elliptic curve defined by $y^2 = x^3 + Ax + B$.
- The 2 division field of an elliptic curve is the field containing the roots of $x^3 + Ax + B$.



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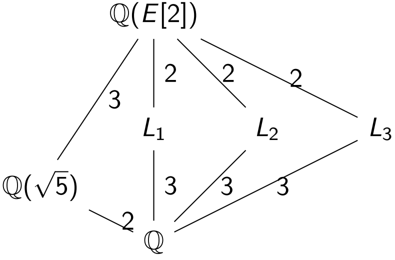
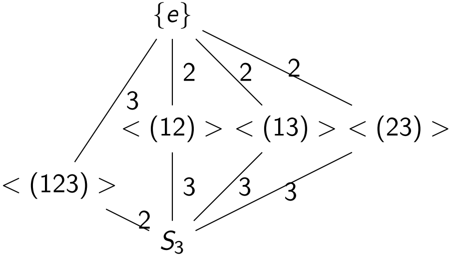
  3. If all the roots are irrational and Δ_E is a perfect square, then the corresponding Galois group is C_3.
  4. If all the roots are irrational and Δ_E is not a perfect square, then the corresponding Galois group S_3.

2 Division Fields Becoming Abelian

Theorem

If the 2 division field is an S_3 extension over $\mathbb{Q}$, then the 2 division field is abelian over $\mathbb{Q}(\sqrt{5})$ iff $\Delta_E = 5d$, where d is a perfect square.





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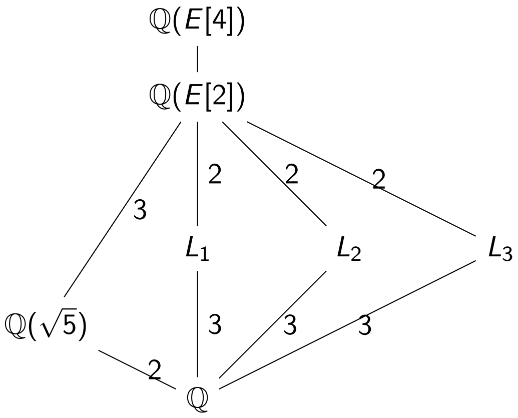

Sub groups of $Gal(\mathbb{Q}(E[3])/\mathbb{Q})$

Theorem (Zywina, 2015)

$Gal(\mathbb{Q}(E[3])/\mathbb{Q})$ is isomorphic to one of the following subgroups of $GL_2(\mathbb{F}_3)$: $C_2, D_4, D_6, SD_{16}, S_3$, and $GL_2(\mathbb{F}_3)$.

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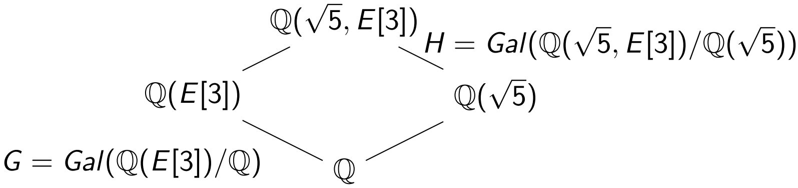
Conjecture

If $Gal(\mathbb{Q}(E[3])/\mathbb{Q})$ is not abelian, then $Gal(\mathbb{Q}(E[3])/\mathbb{Q}(\sqrt{5}))$ is not abelian.

When $\mathbb{Q}(E[3])$ and $\mathbb{Q}(\sqrt{5})$ are Linearly Disjoint

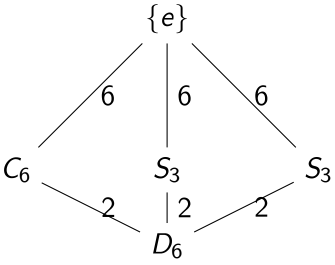


If $\mathbb{Q}(E[3])$ and $\mathbb{Q}(\sqrt{5})$ are linearly disjoint (i.e. $\mathbb{Q}(E[3]) \cap \mathbb{Q}(\sqrt{5}) = \mathbb{Q}$), then $H \cong G$. However, being not disjoint with $\mathbb{Q}(\sqrt{5})$, then it already contains $\mathbb{Q}(\sqrt{5})$ when you base change.

$Gal(\mathbb{Q}(E[3])/\mathbb{Q}) \cong D_6$

Proposition

The three subgroups of order six inside of D_6 are S_3, S_3, and C_6, which is the only abelian subgroup.

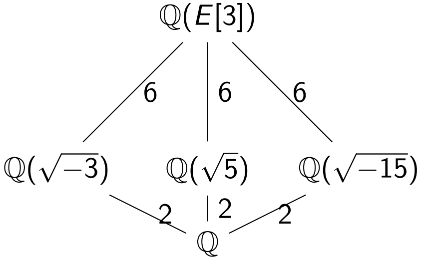


$Gal(\mathbb{Q}(E[3])/\mathbb{Q}) \cong D_6$

Theorem

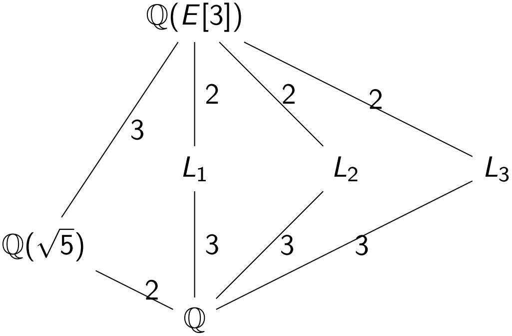
If $Gal(\mathbb{Q}(E[3])/\mathbb{Q}) \cong D_6$, then $Gal(\mathbb{Q}(E[3])/\mathbb{Q}(\sqrt{5}))$ is also nonabelian.

When we look at the subgroup of D_6, we consider the corresponding matrices in $GL_2(\mathbb{F}_3)$. The determinants of the matrices that correspond to C_6 are always $1 \mod 3$ and, hence, correspond to the extension $\mathbb{Q}(\sqrt{-3})$.



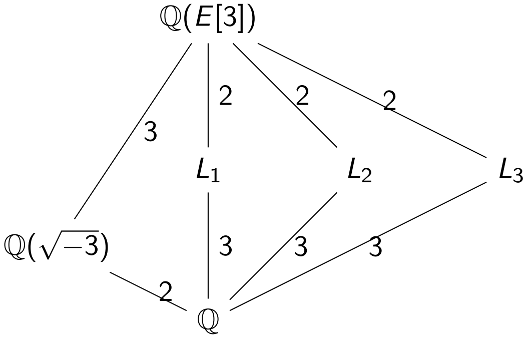
$Gal(\mathbb{Q}(E[3])/\mathbb{Q}) \cong S_3$

Consider that in order for $Gal(\mathbb{Q}(E[3])/\mathbb{Q}(\sqrt{5}))$ to be abelian, then $\mathbb{Q}(E[3])$ must contain $\mathbb{Q}(\sqrt{5})$ as depicted in the following diagram:



$Gal(\mathbb{Q}(E[3])/\mathbb{Q}) \cong S_3$

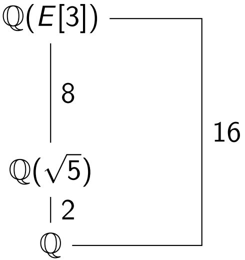
However, we know that this can never be the case because $\mathbb{Q}(\sqrt{-3})$ is the unique quadratic field in $\mathbb{Q}(E[3])$ since $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3) \subseteq \mathbb{Q}(E[3])$ and $\mathbb{Q}(\sqrt{-3})$ is degree 2 over $\mathbb{Q}$.



$Gal(\mathbb{Q}(E[3])/\mathbb{Q}) \cong SD_{16}$

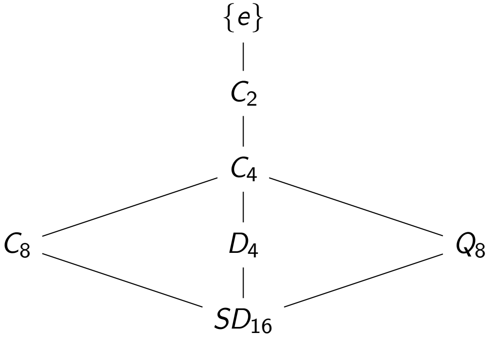
Proposition

Since $\mathbb{Q}(\sqrt{5})$ is a degree two extension, the corresponding field must have relative Galois group $Gal(\mathbb{Q}(E[3])/\mathbb{Q}(\sqrt{5}))$ that has order $16/2 = 8$. This is because SD_{16}, which is not abelian over $\mathbb{Q}$, has order 16.



$Gal(\mathbb{Q}(E[3])/\mathbb{Q}) \cong SD_{16}$

Up to isomorphism, there are only two subgroups of order eight in SD_{16}, namely Q_8 and C_8. C_8 is abelian and the nonsplit Cartan subgroup of $GL_2(\mathbb{F}_3)$. That nonsplit Cartan subgroup is the relative Galois group of an imaginary quadratic extension inside of $\mathbb{Q}(E[3])$.



$Gal(\mathbb{Q}(E[3])/\mathbb{Q}) \cong SD_{16}$

Since C_8 corresponds to an imaginary field which is not over $\mathbb{Q}(\sqrt{-3})$ nor $\mathbb{Q}(\sqrt{5})$, then we know that the relative Galois group $Gal(\mathbb{Q}(E[3])/\mathbb{Q}(\sqrt{5}))$ is not isomorphic to C_8.

Theorem

If $Gal(\mathbb{Q}(E[3])/\mathbb{Q}) \cong SD_{16}$, then $Gal(\mathbb{Q}(E[3])/\mathbb{Q}(\sqrt{5}))$ is not abelian.

$Gal(\mathbb{Q}(E[3])/\mathbb{Q}) \cong D_4$

Now to introduce the problem child of the 3rd division field's Galois group, D_4.

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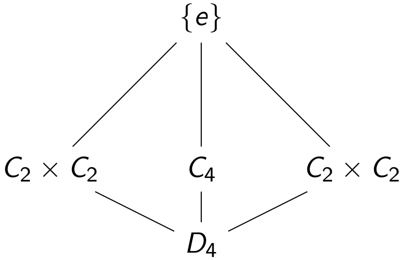
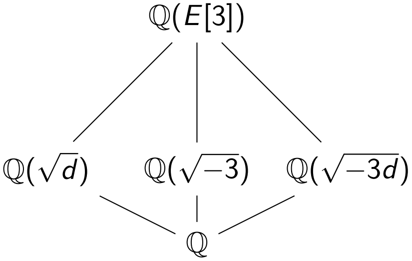


In the diagram we see that D_4 has these three subgroups of index 2 and we want to rule out that $\mathbb{Q}(\sqrt{5})$ could ever be one of the corresponding quadratic fields.

$Gal(\mathbb{Q}(E[3])/\mathbb{Q}) \cong D_4$



BOOO 5! GO HOME

- If 5 does appear then this field diagram would go from non-abelian D_4 to some abelian order 4 group, and every order 4 group is abelian.
- We've worked over 1000's of curves where 5 was of bad reduction and we found no examples, concluding that 5 will never happen.

$Gal(\mathbb{Q}(E[5])/\mathbb{Q})$

A paper of Zywina's classify all possible subgroups of $GL_2(\mathbb{F}_5)$ that appear as the galois group of a 5 division field for some elliptic curve, $E/\mathbb{Q}$.

$Gal(\mathbb{Q}(E[5])/\mathbb{Q})$

- $C_2 \times C_4$
- C_4^2
- OD_{16}
- $C_4 \wr C_2$
- $C_2 \times F_5$
- $C_{24} : C_2$
- $C_4 \times F_5$
- $C_4.S_4$
- C_4
- F_5

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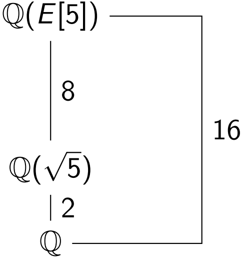
- F_5

$Gal(\mathbb{Q}(E[5])/\mathbb{Q}) \cong OD_{16}$

Proposition

Since $\mathbb{Q}(\sqrt{5})$ is a degree two extension, the corresponding field must have relative Galois group $Gal(\mathbb{Q}(E[5])/\mathbb{Q}(\sqrt{5}))$ that has order $\frac{16}{2} = 8$. This is because OD_{16}, which is not abelian over $\mathbb{Q}$, has order 16.



$Gal(\mathbb{Q}(E[5])/\mathbb{Q}) \cong OD_{16}$

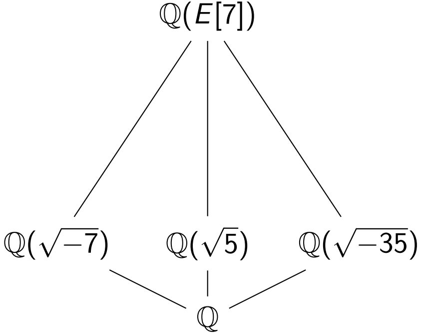
Up to isomorphism, there are only two subgroups of order 8 in OD_{16}, namely $C_4 \times C_2$ and C_8 both of which are abelian.

Theorem

If $Gal(\mathbb{Q}(E[5])/\mathbb{Q}) \cong OD_{16}$, then $Gal(\mathbb{Q}(E[5])/\mathbb{Q}(\sqrt{5}))$ is abelian.

$Gal(\mathbb{Q}(E[7])/\mathbb{Q})$

If $\mathbb{Q}(\sqrt{5}) \subseteq \mathbb{Q}(E[7])/\mathbb{Q}$, then $\mathbb{Q}(E[7])$ has 3 quadratic subfields: $\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{-7}), \mathbb{Q}(\sqrt{-35})$. This means Gal($\mathbb{Q}(E[7])/\mathbb{Q}$) has 3 index 2 subgroups. Since the Galois group is a subgroup of $\mathrm{GL}_2(\mathbb{F}_7)$, we only need to look at subgroups of $\mathrm{GL}_2(\mathbb{F}_7)$ which have at least 3 index 2 subgroups.



$Gal(\mathbb{Q}(E[7])/\mathbb{Q})$

Group	Order
C_6^2	36
$C_6 \times S_3$	36
$C_2 \times F_7$	84
$C_6 \times D_7$	84
$C_6 \wr C_2$	72
$C_6 \times F_7$	252
$C_3 \times \mathrm{SD}_{32}$	96

Table: Possible Subgroups of $\mathrm{GL}_2(\mathbb{F}_7)$ up to isomorphism

These groups are subgroups of $\mathrm{GL}_2(\mathbb{F}_7)$ that can appear as Galois groups of 7-division fields over $\mathbb{Q}$ that have at least 3 subgroups of index 2.

$Gal(\mathbb{Q}(E[7])/\mathbb{Q})$

Group	Order
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Table: Possible Subgroups of $\mathrm{GL}_2(\mathbb{F}_7)$ up to isomorphism

These groups are subgroups of $\mathrm{GL}_2(\mathbb{F}_7)$ that can appear as Galois groups of 7-division fields over $\mathbb{Q}$ that have at least 3 subgroups of index 2.

So far, the potential groups with index 2 subgroups that could become abelian after a base change to $\mathbb{Q}(\sqrt{5})$ are: $C_6 \times S_3$, $C_6 \wr C_2$, and $C_3 \times \mathrm{SD}_{32}$.

Future Works

- We plan to finish our work in 3 and 7 division fields. We also plan to follow up on some promising results in 4 and 10 division fields.
- We would like to determine which other primes p and composites n can give us abelian extensions over $\mathbb{Q}(\sqrt{5})$.
- We would like to look at division fields of elliptic curves that are defined over $\mathbb{Q}(\sqrt{5})$ and not over $\mathbb{Q}$.
- We would also like to extend this work to CM elliptic curves as well.

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